Performance of a Normal Process Distribution

***MahendraSaha**

*Assistant Professor, Department of statistics, Central University of rajasthan, Bandarsindri, Ajmer, 305 817, Rajasthan

Abstract

Generalized process capability index, defined as the ratio of proportion of specification conformance to proportion of desired conformance, has been given by Maiti et al. (2010). Normal process has been taken into account. Under this distributional assumption, small sample as well as large sample properties of point estimators of the generalized process capability index for centered, off-centered and off-target process has been made. A data set has been analysed.

Keywords: Control chart, normal distribution, process yield, process capability index.

1. Introduction

Quantifying the "capability" of a manufacturing process is an important initial step in any quality improvement program. Capability is usually defined in dictionaries as "the ability to carry out a task, to achieve an objective". PCIs which establish the relationship between the actual process performance and the manufacturing specifications have been a focus of research in quality assurance and process capability analysis. Generalized Process Capability Index, defined as the ratio of proportion of specification conformance (or, process yield) to proportion of desired (or, natural) conformance. Further development of generalized process capability index for off-centered and off-target process has been made.

The article is organized as follows. We give a brief review on the PCIsC_p, C_{pk}, C_{pm} and C_{pmk} in section 2. In sections 3 and4, we derive the point estimators (MLE and MVUE) for generalized process capability index (with process median being the process center) under the assumption of normal process distribution and simulation results have been reported and discussed. In section5, a data set has been analyzed to demonstrate the application of the generalized process capability index. Section 6concludes.

2. Background

The most popular PCIs areC_p, C_{pk}, C_{pm} and C_{pmk} .The $C_p$index is defined as

$$C_p = \frac{U-L}{6\sigma},$$

where $L$and $U$are the lower and upper specification limits, respectively, and $\sigma$is the process standard deviation. Note that C_p does not depends on the process mean.The $C_{pk}$is then introduced to reflect the impact of μon the process capability indices. The C_{pk}index is defined as

$$C_{pk} = \min\left\{\frac{U-\mu}{3\sigma}, \frac{\mu-L}{3\sigma}\right\}.$$

The C_{pm}index was introduced by Chan et al. (1988). This index takes into account the influence of the departure of the process mean $\mu$from the process target$T$.The C_{pm}index is defined as

$$C_{pm} = \frac{U-L}{\sqrt{\sigma^2 + (\mu - T)^2}}.$$

The measure C_{pm} sometimes called the "Taguchi index". There is also the hybrid index given by Pearn et al. (1992),

$$C_{pmk} = \min\left\{\frac{U-\mu}{3\sqrt{\sigma^2+(\mu-T)^2}}, \frac{\mu-L}{3\sqrt{\sigma^2+(\mu-T)^2}}\right\}$$

$$= \frac{d-|\mu-M|}{3\sqrt{E(\mu-T)^2}}.$$

Clearly, C_{pmk} is only meaningful when deviation from target is the main concern. This index is based on the quadratic loss function and, thus, should only be used when there is evidence of a quadratic monetary loss. There are an infinite number of possible loss functions, and in many practical cases, material outside specification limits may result in a total loss rather than a quadratic loss. Clearly $C_p \geq C_{pk} \geq C_{pmk}$ and $C_p \geq C_{pm} \geq C_{pmk}$.

3. Generalized Process Capability Index

Process Capability Indices (PCIs) aim to quantify the capability of a process of quality characteristic (X) to meet some specifications that are related to a measurable characteristic of its produced items. It appears to be a general acceptance of the idea that PCIs can be used only after it has been established that a process is in "statistical control" (for example, by the use of control charts).A generalized Process Capability Index, defined as the ratio of proportion of specification conformance (or, process yield) to proportion of desired (or, natural) conformance. Almost all the process capabilities defined in the literature are directly or indirectly associated with this generalized index. Normal as well as non-normal and continuous as well as discrete random variables could be covered by this index. It can also be assessed under either unilateral or bilateral specifications.The generalized process capability index is defined as

$$C_{py} = \frac{p}{p_0}. \qquad (3.1)$$

Where,$p = F(U) - F(L)$and $p_0 = F(UDL) - F(LDL)$. Practitioners may realize these limits as lower tolerance limit (LTL) and upper tolerance limit (UTL) respectively. When the process is off-centered, then$F(L) + F(U) \neq 1$, but the proportion of desired conformance achieved. In that case the index is as follows:

$$C_{pyk} = \min\{C_{pyu}, C_{pyl}\}, \qquad (3.2)$$

where, $$C_{pyu} = \frac{F(U)-0.5}{0.5-\alpha 2}$$

and $$C_{pyl} = \frac{0.5-F(L)}{0.5-\alpha_1},$$

with$\mu_e$being the median of the distribution and the process center is to be located such that$F(\mu_e) = \{F(L) + F(L)\}/2$, i.e., $F(L) + F(U) = 1$, $\alpha_1 = P(X < LDL)$ and $\alpha_2 = P(U > UDL)$. It generally happens that process target T is such that$F(T) = \{F(L) + F(U)\}/2$; if $F(T) \neq \{F(L)F(U)\}/2$, the situation may be described as "generalized asymmetric tolerances" have described by the term "asymmetric tolerances" when $T \neq (L + U)/2$.Under this circumstance, the index is defined as follows:

$$C_{pTk} = \min\{C_{pTu}, C_{pTl}\}, \quad (3.3)$$

where, $$C_{pTu} = \frac{F(U) - F(T)}{0.5 - \alpha_2}$$

and $$C_{pTl} = \frac{F(T) - F(L)}{0.5 - \alpha_1}.$$

This generalized process capability index that could be used for normal as well as discrete quality characteristics. Almost all the most widely used capability indices are directly or indirectly associated with this index. It could be used comfortably by the practitioners.

4. Generalized Process Capability Index

This section deals with the generalized process capability index (GPCI) for normal process. Inference about index C_{py} is equivalent to that about $p = F(U) - F(L)$.

Assuming normality of the examined process, its yield is given by

$$p = F(U) - F(L)$$

$$= \Phi\left(\frac{U-\mu}{\sigma}\right) - \Phi\left(\frac{L-\mu}{\sigma}\right),$$

where μ and σ denote the mean and the standard deviation of the process respectively, and $\Phi(.)$ is the cumulative distribution function of the standard normal variate. The value of p can be estimated using the estimator

$$\hat{p} = \Phi\left(\frac{U-\bar{x}}{s}\right) - \Phi\left(\frac{L-\bar{x}}{s}\right),$$

where $\bar{x}$ and s are the sample mean and the sample standard deviation respectively, obtained from a random sample of size n. Hence the maximum likelihood estimator (MLE) of C_{py} is

$$\hat{C}_{py} = \frac{\hat{p}}{p_0}. \quad (4.4)$$

Using Lehmann-Scheffe theorem, the minimum variance unbiased estimator of p can be obtained as

$$\tilde{p} = \int_{\sqrt{n}\frac{(L-\bar{X})}{(n-1)s}}^{\sqrt{n}\frac{(U-\bar{X})}{(n-1)s}} \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-2}{2})}(1-u^2)^{\frac{n-4}{2}}\,du$$

since the conditional distribution of X given $(\bar{X},\ s)$ is

$$f(x/\bar{X}, s) = \frac{\sqrt{n}\,\Gamma(\frac{n-1}{2})}{\sqrt{\pi}(n-1)s\Gamma(\frac{n-2}{2})}\left[1 - \frac{n(x-\bar{X})^2}{(n-1)^2 s^2}\right]^{\frac{n-4}{2}},$$

for $\bar{X} - s(n-1) < x < \bar{X} + s(n-1)$. Hence the minimum variance unbiased estimator (MVUE) of C_{py} is given as

$$\tilde{C}_{py} = \frac{\tilde{p}}{p_0}. \quad (4.5)$$

Now we carry out simulation study to compare the estimators (MLE and MVUE) of C_{py} using the equation (4.4) and (4.5).The estimators and their Mean Square Errors (MSEs) are presented in the following table for $p_0 = 0.90$ and for choice of (L, U)as (0,10)and for sample sizes $n = 25, 50,$ 100, 150 and 200. 25,000such repetitions are made to calculate the average $\hat{C}_{py}$ and average $\tilde{C}_{py}$ and their MSEs. We generate observations from normal distributions with choices of$(\mu, \sigma) =$ (5,3), (5,4), (6,3)and (6,4). First column of the table show the values of $\mu, \sigma$and the correspondingC_{py}. Remaining columns show average $\hat{C}_{py}$ and its MSE and average $\tilde{C}_{py}$ and its MSE, respectively, for the above mentioned sample sizes. It is observed that in all most all the cases MSE of $\hat{C}_{py}$ is less than that of$\tilde{C}_{py}$. As expected, MSEs are almost equal for large sample sizes.

Table 1: Estimates of C_{py} and their MSEs with $L = 0$, $U = 10$, samples generated from normal distribution.

μ, σ, C_{py} / n	25	50	100	150	200
(5,3)	1.005517738	1.005178344	1.0049998458	1.0048103289	1.0048546901
	0.002426411	0.001274405	0.0006740702	0.0004459613	0.0003414191
1.004910328	1.012077716	1.008466977	1.0066411223	1.0059034939	1.0056743211
	0.002566664	0.001313873	0.0006846556	0.0004503547	0.0003439839
(5,4)	0.882290039	0.879793233	0.878030195	0.8773517235	0.8771119184
	0.004884550	0.002489539	0.001274883	0.0008398995	0.0006401974
0.876333836	0.885645657	0.881357107	0.878778235	0.8778422070	0.8774770802
	0.005208379	0.002572843	0.001295943	0.0008490264	0.0006454093
(6,3)	0.986129458	0.985330398	0.9849121747	0.9847368536	0.9848391261
	0.002995628	0.001535556	0.0007876467	0.0005331396	0.0003963808
0.984487387	0.991619397	0.988086222	0.9862870282	0.9856533961	0.9855266060
	0.003163933	0.001580817	0.0007993607	0.0005383535	0.0003995182
(6,4)	0.864134984	0.862062126	0.862290146	0.8617021225	0.8614464514
	0.002618593	0.001317132	0.001335275	0.0008751462	0.0006572730
0.860597272	0.865594175	0.862762209	0.862991604	0.8621639133	0.8617905439
	0.002690406	0.001334610	0.001353302	0.0008829455	0.0006616727

In case of off-centered situation (i.e., when$F(L) + F(U) \neq 1$), we have already defined

$$C_{pyk} = \min\left[\frac{p_u - 0.5}{0.5 - \alpha_2}, \frac{0.5 - p_l}{0.5 - \alpha_1}\right]$$

Where, $p_u = P(X < U)$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{U-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{U-\mu}{\sigma}\right),$$

and$p_l = P(X < L)$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{L-\mu}{\sigma}\right)$$
$$= \Phi\left(\frac{L-\mu}{\sigma}\right)$$

Obtaining $\bar{x}$ands, the sample mean and the samplestandard deviation respectively from a random sample ofsize $n$to find out the MLE of $p_u$andp_l, which are given as

$$\hat{p}_u = \Phi\left(\frac{U-\bar{x}}{s}\right)$$

and

$$\hat{P}_l = \Phi\left(\frac{L-\bar{x}}{s}\right)$$

and using the invariance property of MLE, we can easily find out the MLE of C_{pyk} as

$$\hat{C}_{pyk} = \min\left[\frac{\hat{p}_u - 0.5}{\frac{1}{2}-\alpha_2}, \frac{0.5-\hat{p}_l}{\frac{1}{2}-\alpha_1}\right].$$

To find out the MVUE ofC_{pyk}, at first we have to find out the MVUE of $p_u$and $p_l$respectively, which are given as

$$\tilde{p}_u = \int_0^U \frac{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}{(n-1)s\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}\left[1-\frac{n(x-\bar{X})^2}{(n-1)^2 s^2}\right]^{\frac{n-4}{2}} dx$$

and

$$\tilde{p}_l = \int_0^L \frac{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}{(n-1)s\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}\left[1-\frac{n(x-\bar{X})^2}{(n-1)^2 s^2}\right]^{\frac{n-4}{2}} dx\,.$$

Then, we can find a plug-in estimator $\hat{C}_{pyk}$ of C_{pyk} which is given by

$$\hat{C}_{pyk} = \min\left[\frac{\tilde{p}_u - 0.5}{0.5-\alpha_2}, \frac{0.5-\tilde{p}_l}{0.5-\alpha_1}\right].$$

We simulate observations from normal distribution to compare the estimators (MLE and MVUE) ofC_{pyk}. We take$L = 0$, $U = 8$,$\alpha_1 = 0.04$and$\alpha_2 = 0.06$, all other set-ups remain same as discussed earlier. Here also we see that in all most all the cases MSE of $\hat{C}_{pyk}$ is less than that of $\hat{C}_{pyk}$.

Table 2: Estimates of C_{pyk} and their MSEs with $L = 0$, $U = 8$, $\alpha_1 = 0.04$ and $\alpha_2 = 0.06$, samples generated from normal distribution.

μ, σ, C_{pyk} / n	25	50	100	150	200
(5,3)	0.78617494	0.780697462	0.778734143	0.778473520	0.777044558
	0.01781331	0.008865522	0.004535656	0.002978109	0.002239725
0.775783514	0.68534603	0.675723490	0.672055263	0.671081344	0.669491700
	0.02947563	0.020622312	0.016156349	0.014463103	0.013944571
(5,4)	0.62681401	0.62709551	0.624634261	0.623554861	0.623459508
	0.01205039	0.01215308	0.006006254	0.003986055	0.003050909
0.62130147	0.39118284	0.39156336	0.386477961	0.384926554	0.384713053
	0.06520912	0.06506308	0.061305283	0.059906070	0.059057351
(6,3)	0.57549463	0.56963948	0.566093993	0.564284201	0.564532433
	0.02640220	0.01307163	0.006446083	0.004308632	0.003239196
0.56251696	0.52584932	0.51924624	0.514787919	0.512925664	0.513165166
	0.02758598	0.01476357	0.008618120	0.006682562	0.005599852
(6,4)	0.44679518	0.44126404	0.43761329	0.43729474	0.436970849
	0.03049218	0.01500324	0.00736355	0.00489519	0.003641562
0.43514196	0.29972101	0.29234175	0.28748604	0.28629944	0.285644087
	0.04456109	0.03308494	0.02803485	0.02632046	0.025400406

In case of off-target situation, using the invariance property of MLE, we can find out the MLE of the index C_{pTk}, given as

$$\hat{C}_{pTk} = \min\left[\frac{\hat{p}_{tu}}{0.5-\alpha_2}, \frac{\hat{p}_{tl}}{0.5-\alpha_1}\right]$$

where, $$\hat{p}_{tu} = \Phi\left(\frac{U-\bar{x}}{s}\right) - \Phi\left(\frac{T-\bar{x}}{s}\right)$$

and $$\hat{p}_{tl} = \Phi\left(\frac{T-\bar{x}}{s}\right) - \Phi\left(\frac{L-\bar{x}}{s}\right).$$

To find out the MVUE of C_{pTk} , here also at first we have to find out the MVUE of p_{tu} and p_{tl} respectively, which are given as

$$\tilde{p}_{tu} = \int_T^U \frac{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}{(n-1)s\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}\left[1-\frac{n(x-\bar{X})^2}{(n-1)^2 s^2}\right]^{\frac{n-4}{2}} dx$$

and

$$\tilde{p}_{tl} = \int_L^T \frac{\sqrt{n}\,\Gamma\left(\frac{n-1}{2}\right)}{(n-1)s\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}\left[1-\frac{n(x-\bar{X})^2}{(n-1)^2 s^2}\right]^{\frac{n-4}{2}} dx.$$

And we can find a plug-in estimator $\widehat{C}_{pTk}$ of C_{pTk} which is given by

$$\widehat{C}_{pTk} = \min\left[\frac{\tilde{p}_{tu}}{0.5-\alpha_2}, \frac{\tilde{p}_{tl}}{0.5-\alpha_1}\right].$$

Again we carry out a simulation study to compare the estimators of C_{pTk}. All other set up remain same as discussed earlier. In all cases MSE of $\hat{C}_{pTk}$ is less than that of $\widehat{C}_{pTk}$. As we enter into the unbiased class, we are losing some efficiency. It is to be noticed that MLE is not an unbiased estimator. To find MVUE one has to perform numerical method of integration. For this complexity, it may be concluded that the use of MLE sometimes fairly adequate.

Table 3: Estimates of C_{pTk} and their MSEs with $L = 0$, $U = 8$, $\alpha_1 = 0.04$ and $\alpha_2 = 0.06$, samples generated from normal distribution.

μ, σ, C_{pTk} / n	25	50	100	150	200
(5,3)	0.78617494	0.780697462	0.778734143	0.778473520	0.777044558
	0.01781331	0.008865522	0.004535656	0.002978109	0.002239725
0.775783514	0.68534603	0.675723490	0.672055263	0.671081344	0.669491700
	0.02947563	0.020622312	0.016156349	0.014463103	0.013944571
(5,4)	0.62681401	0.62709551	0.624634261$	00.623554861	0.623459508
	0.01205039	0.01215308	0.006006254	00.003986055	0.003050909
0.62130147	0.39118284	0.39156336	0.386477961	0.384926554	0.384713053
	0.06520912	0.06506308	0.061305283	0.059906070	0.059057351
(6,3)	0.57549463	0.56963948	00.566093993	0.564284201	0.564532433
	0.02640220	0.01307163	0.006446083	0.004308632	0.003239196
0.56251696	0.52584932	0.51924624	0.514787919	00.512925664	0.513165166
	0.02758598	0.01476357	0.008618120	0.006682562	0.005599852
(6,4)	0.44679518	0.44126404	0.43761329	0.43729474	0.436970849
	0.03049218	0.01500324	0.00736355	0.00489519	0.003641562
0.43514196	0.29972101	0.29234175	0.28748604	0.28629944	0.285644087
	0.04456109	0.03308494	0.02803485	0.02632046	0.025400406

5. Real-world Application

This section is devoted to the inferential aspect of the generalized process capability index by analyzing a data set. Here, we provide an example using real life data set [c. f. Peng(2010)]. Since the parametric modelling is heavily dependent on the correct model specification, we use the Kolmogrov-Smirnov (K-S) goodness of fit test and correlation coefficient value in q-q plotting to identify the best parametric model to fit the underlying process.

Voltage of Aluminium Foil:

A capacitor built in a circuit may determine the resonant frequency and quality factor of a resonant circuit, power dissipation and operating frequency in a digital logic circuit, energy capacity in a high-power system and many other aspects. Aluminium foil is one of the crucial components that determine the quality of a capacitor. The voltage is an important quality characteristic of the aluminium foil. The production specifications of the voltage are $(L, T, U) = (510,520,530)$. Calculation of generalized capability index boils down to calculation of the process yield. To calculate the process yield, it necessitates to apply a curve fitting method to approximate the quality characteristic distribution, $f(x)$. The data set approximately fit to normal distribution with mean 522.172 and standard deviation 2.974 for which p value of K-S test is 0.2807 and correlation of the q-q plotting is 0.995. Histogram and fitted curve to normal distribution and control of this data set has been shown in the following figure. All the calculated indices are presented in Table 4.

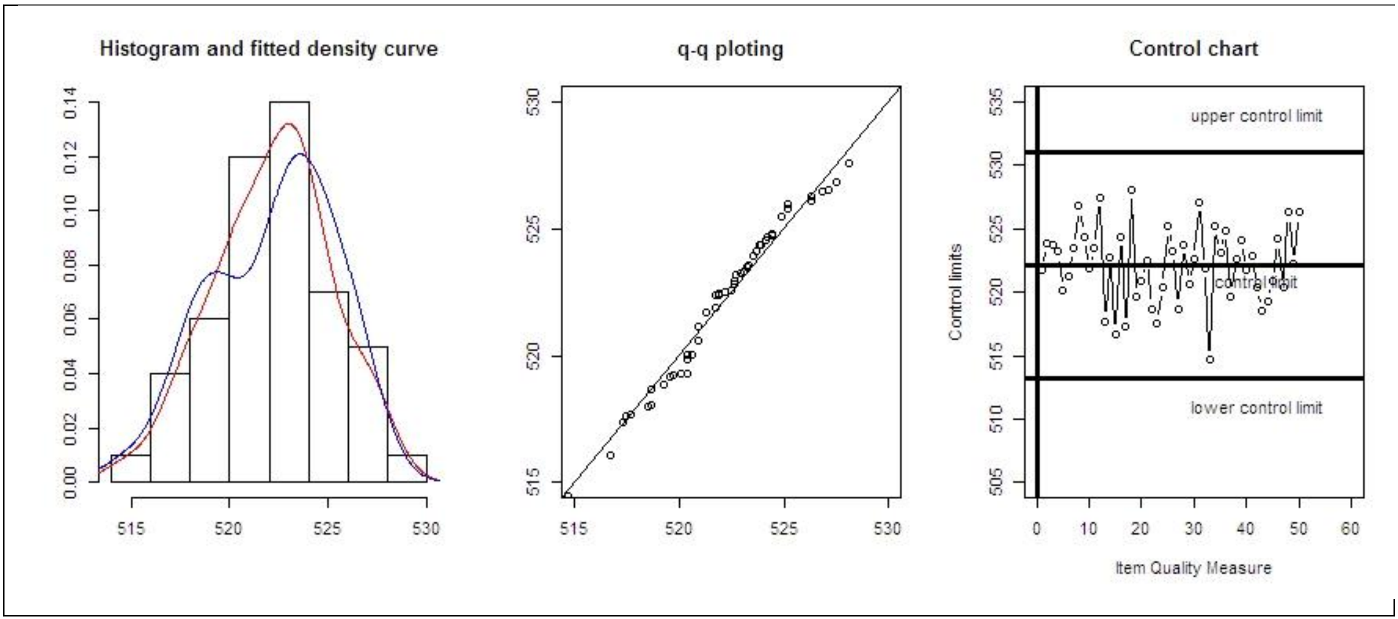

Figure 1: Histogram and fitted density curve, q-q plotting and control chart of the data

Table 4: Different Process Capability Indices calculated using data in Peng (2010).

$C_p = 1.120681$	$C_{pk} = 0.8772687$	$C_{pm} = 0.7084791$
$C_{py} = 0.9984278$	$C_{pyk} = 0.9941911$	$C_{pTk} = 0.466463$

6. Conclusions

In this article the inferential aspects of generalized process capability index has been presented. The MLE and MVUE of this generalized process capability index have been studied for normal process distribution and the estimators are compared through simulation study in different situations. The index is easy and comfortable to practitioners as well as interesting to the theoreticians.

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